Arithmetic Expression Geometry: Five Possibilities

Mingli Yuan

October 16, 2025

Table of Contents

Foreword: A few small, colorful stones

Over the past ten years of exploration, I've collected a few small, colorful stones.

Each one sparked my curiosity, and together they have carried me to where I am today.

The stones are not answers, but invitations — to keep exploring, to keep asking, and to enjoy the process.

Asking the right questions is often more important than finding the right answers, especially in the AI era.

From Non-commutativity to Complexity

How does non-commutativity lead to complexity, and how can we geometrize it?

A well-known example is word2vec:

- Parallelism encodes semantic analogy: concepts are points, relations are vectors.
- However, this regularity is not fully or rigorously enforced in word2vec. What happens if we enforce it strictly?

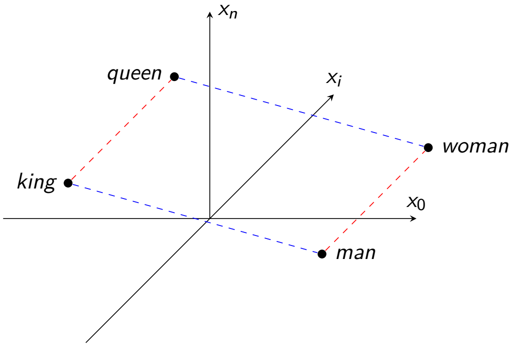
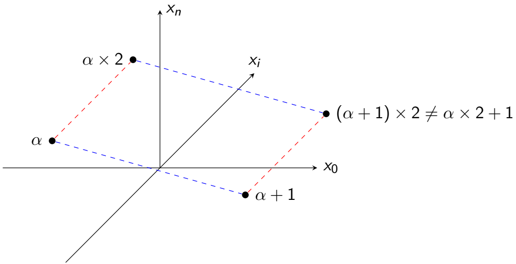


Figure: Regularity in word2vec

$$(\alpha + 1) \times 2 \neq \alpha \times 2 + 1$$

- First-class elements: numbers (points), operations (generators), and relationships (operation sequences/paths) are modeled explicitly.
- Regularity strictly enforced: the same operation sequence induces the same geometric displacement; different orders encode different relationships.
- Both "relationship" and "concept" are treated as first-class, fully consistent with this regularity.



- Once non-commutativity is introduced, the number of distinct operation strings grows *exponentially*.
- Generation trees visualize this explosion: greater depth leads to a combinatorial surge in distinct expressions.
- Intuition: branching factor > 1 implies exponential growth.

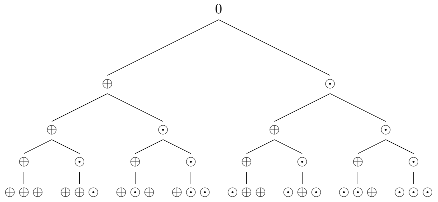


Figure: Tree expansion of possible generation sequences

From symbols to geometry: hyperbolicity and volume

- Assume each operation sequence maps to a point in a geometric space, and that balls of radius δ have uniformly positive volume.
- Then the symbolic explosion caused by non-commutativity corresponds to exponential growth of volume.
- Hyperbolic spaces exemplify this behavior:

$$\mathrm{Vol}(B_r) \sim e^{(n-1)r}$$

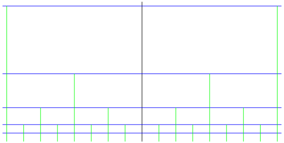


Figure: Non-parallelogram $\Rightarrow$ hyperbolicity

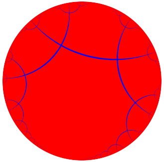


Figure: A hyperbolic tessellation

Contrast: the linear/commutative case

- If operations commute or are constrained by linearity, the *parallelogram law* holds.
- The parallelogram law simplifies the generation process and mitigates the explosion of symbolic combinations.
- Euclidean ball volume: $\mathrm{Vol}(B_r) \propto r^n$ (polynomial in r).

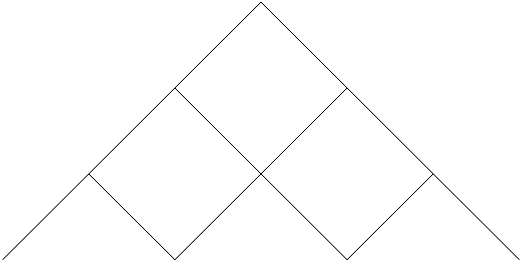


Figure: Parallelogram law $\Rightarrow$ Euclidean linearity

Conclusion: complexity as a geometric volume

We have shown that:

- Commutativity implies the parallelogram law and Euclidean linearity.
- In the commutative (linear) case, volumes grow polynomially: Euclidean space.
- Non-commutativity breaks the parallelogram law, yielding exponential growth: hyperbolic space.

More broadly, geometric volume captures the complexity of operation sequences.

Where Computation, Geometry, and Analysis Meet

Is there a unified framework that brings computation, geometry, and analysis together?

The $\mathfrak{E}_1$ Space: A Perfect Example

In the $\mathfrak{E}_1$ space, computation, geometry, and analysis coincide:

Computational Aspect

The assignment $a = -x/y$ encodes arithmetic expressions as a geometric flow.

Geometric Aspect

The space has a hyperbolic metric $ds^2 = \frac{1}{y^2}(\frac{dx^2}{\mu^2} + \frac{dy^2}{\lambda^2})$.

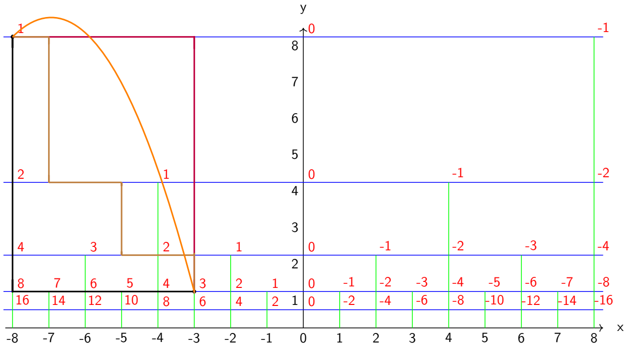
Analytic Aspect

The assignment satisfies the flow equation:

$$\frac{da}{ds} = \mu \cos\theta + \lambda a \sin\theta$$

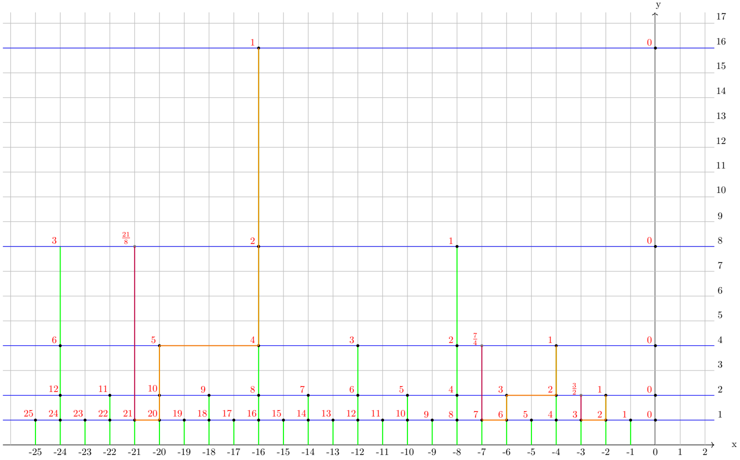
This triad suggests that AEG can serve as a Rosetta Stone for translating among these domains.

- Black path: $1 \times 8 - 5 = 3$
- Purple path: $\left(1 - \frac{5}{8}\right) \times 8 = 3$
- Orange path: an example integral curve



Examples:

$$\left(\frac{3}{2}\right)_{10} = 1.1_2,\ \left(\frac{7}{4}\right)_{10} = 1.11_2,\ \left(\frac{21}{8}\right)_{10} = 10.101_2.$$



Which complexity does volume measure?

It is well known that time and space complexity can trade off against one another.

- Time complexity: number of operations (steps)
- Space complexity: memory usage (intermediate results)

This trade-off suggests a common source of complexity. Which notion of complexity does geometric volume quantify?

Non-commutativity leads to arithmetic torsion, which measures the discrepancy between different operation orders. We often denote torsions as $\tau(p)$ where p is a path.

τ for two operations:

$$(x+1)\times 2-(x\times 2+1)=1$$

τ for three operations:

$$((x+1)\times 2+3)-((x+3)\times 2+1)=-2$$

Arithmetic torsion at scale

As the step size increases:
For a single step:

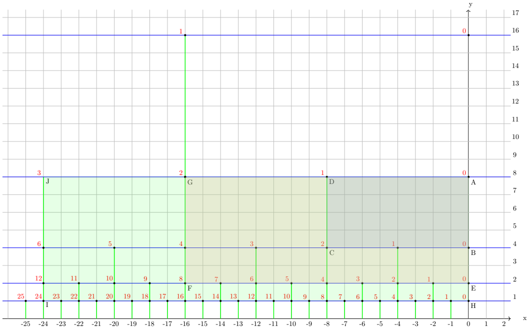
$$(x+1)\times 2-(x\times 2+1)=1 \qquad (1)$$

For two steps:

$$(x+2)\times 4-(x\times 4+2)=6 \qquad (2)$$

For three steps, the pattern continues:

$$(x+3)\times 8-(x\times 8+3)=21 \qquad (3)$$



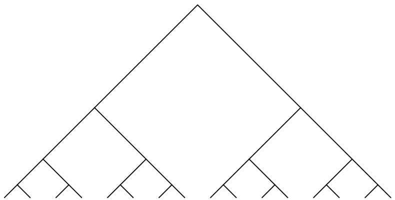


Figure: smaller torsion

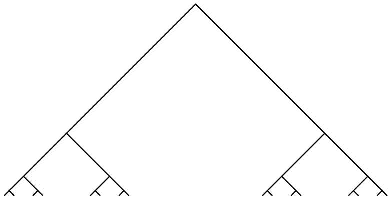


Figure: larger torsion

Grids and Cayley graphs

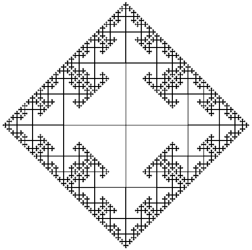


Figure: Free group $< a, b >$

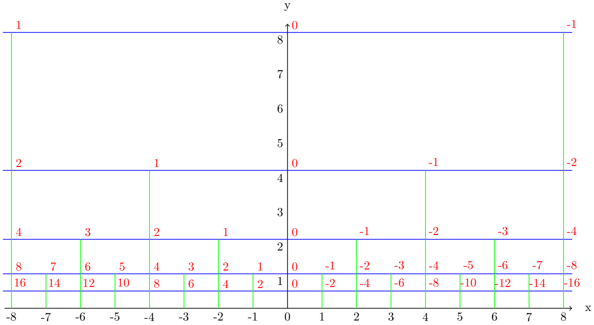
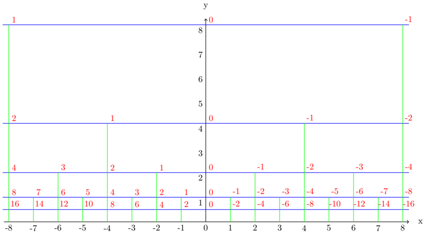
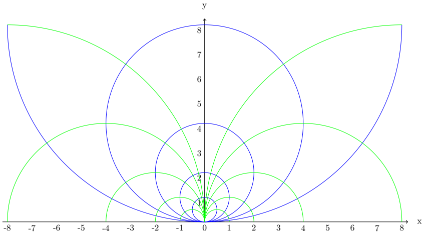


Figure: Baumslag–Solitar group $< a, b | bab^{-1} = a^2 >$

One space, two grids and conformal map





$$z \mapsto -\frac{1}{z}$$

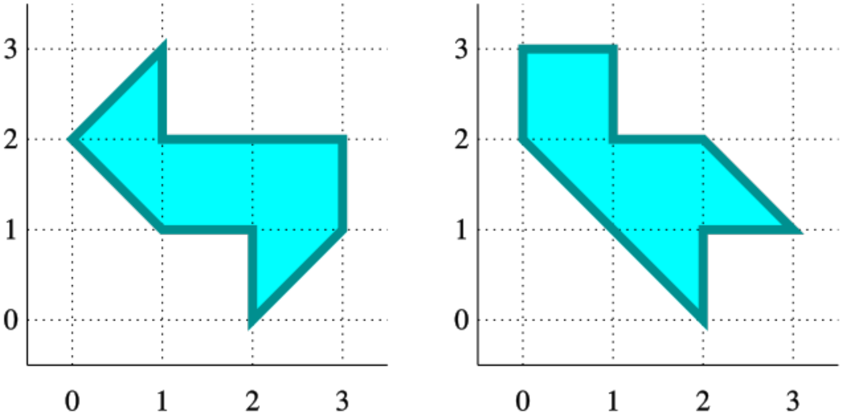


Figure: Isospectral drums

In this setting, $A = \frac{1}{\mu y}$ and $B = \frac{1}{\lambda y}$:

$$\Delta f = y^2 \left(\mu^2 \frac{\partial^2 f}{\partial x^2} + \lambda^2 \frac{\partial^2 f}{\partial y^2} \right)$$

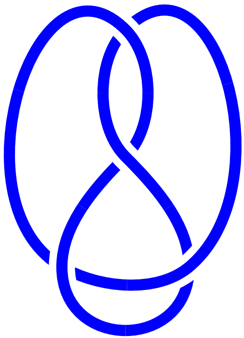
For $f = -\frac{x}{y}$, it follows that

$$\Delta f = -\frac{2\lambda^2 x}{y} = 2\lambda^2 f$$

Therefore, $f = -\frac{x}{y}$ is an eigenfunction of the Laplacian with eigenvalue $2\lambda^2$.

The knot group of the figure-eight knot (4_1) has the presentation:

- Generators: a, b
- Relator: $abbbaBAAB = 1$ (where $A = a^{-1}, B = b^{-1}$)
- Alexander polynomial: $\Delta(t) = t^2 - 3t + 1$



$$
\begin{aligned}
x &= a(b(b(b(a(B(A(A(B(x))))))))) \\
&= a(b(b(b(a(B(A(A(x-1)))))))) \\
&= a(b(b(b(a(B(A((x-1)t^{-1}))))))) \\
&= a(b(b(b(a(B((x-1)t^{-2})))))) \\
&= \cdots \\
&= x-(t^2-3t+1)
\end{aligned}
$$

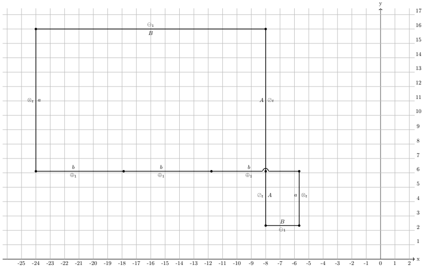
$$x-(t^2-3t+1) = x$$

This forces the condition $t^2 - 3t + 1 = 0$

Cyclotomic polynomials appear in the global arithmetic torsion of the AEG path;

$$\tau(p) = \frac{-\Delta_{4_1}(a)(a^2-1)}{a^2}$$

.



Conclusion: richness from the unified perspective

We have shown that:

- Arithmetic expressions can be encoded as geometric flows in the hyperbolic model $\mathfrak{E}_1$; the assignment $a = -x/y$ links computation, geometry, and analysis via the flow equation.
- Non-commutativity manifests as arithmetic torsion that scales with step size and organizes paths; analytic structure appears through Laplacian eigenfunctions.
- Algebraic invariants such as Alexander and cyclotomic polynomials govern path closure and global torsion, revealing a deep interface between topology and arithmetic.

A Neo-Calculus?

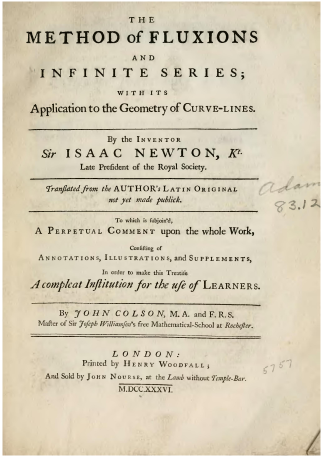
Can we develop a new calculus that naturally handles mixed operations?

How to describe a change over time?

Two complementary ways to describe a small change over time:

- by quantity: add a near-zero amount
- by ratio: multiply by a near-unit factor

Traditional calculus is based on the first method; the Riemann integral is additive. The functions exp and log convert between these two viewpoints.



THE
METHOD of FLUXIONS
AND
INFINITE SERIES;
WITH ITS
Application to the Geometry of CURVE-LINES.

By the INVENTOR
Sir ISAAC NEWTON, *Kt.*
Late Prefident of the Royal Society.

Tranflated from the AUTHOR's LATIN ORIGINAL
not yet made publick.

To which is fubjoin'd,
A PERPETUAL COMMENT upon the whole Work,
Confifting of
ANNOTATIONS, ILLUSTRATIONS, and SUPPLEMENTS,
In order to make this Treatife
A compleat Inftitution for the ufe of LEARNERS.

By *JOHN COLSON*, M.A. and F.R.S.
Mafter of Sir *Jofeph Williamfon*'s free Mathematical-School at *Rochefter*.

LONDON:
Printed by HENRY WOODFALL;
And Sold by JOHN NOURSE, at the *Lamb* without *Temple-Bar*.
M.DCC.XXXVI.

Figure: Method of Fluxions

Figure: Vito Volterra

Matrix-valued non-commutative differentiation and integration, left and right

- $\frac{d}{dx}A(x) = \lim_{\Delta x \to 0} \frac{A(x+\Delta x)A^{-1}(x)-I}{\Delta x}$
- $A(x)\frac{d}{dx} = \lim_{\Delta x \to 0} \frac{A^{-1}(x)A(x+\Delta x)-I}{\Delta x}$
- $\prod_a^b (I + A(x)dx) = \lim_{\nu(P)\to 0} \prod_{i=m}^{1} (I + A(\xi_i))$
- $(I + A(x)dx) \prod_a^b = \lim_{\nu(P)\to 0} \prod_{i=1}^{m} (I + A(\xi_i))$

An identity connects product integration with standard additive integration:

$$\prod_a^b (I + A(x)\,dx) = I + \int_a^b A(x)\,dx + \int_a^b \int_a^x A(x)A(y)\,dy\,dx + \cdots$$

How about mixing up additive and multiplicative steps?

Additive calculus provides a closed algebra of polynomials under differential and integral operators.

- $\frac{d}{dx} p(x) = q(x)$
- $\int p(x)\, dx = q(x) + C$

This underlies power series, Laurent series, and approximation theory.

Problem: How do we mix additive and multiplicative steps in a single process?

Directions:

- We need to handle functions beyond constants in arithmetic expressions.
- We need to expand polynomials into another closed set under differential and integral operators.

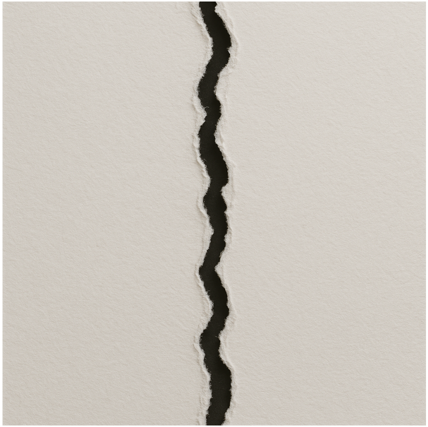
$$
\begin{tikzcd}
x + \mu du \arrow[d, dashed] & & x \arrow[ll, dashed] \arrow[d, dashed] \\
(x + \mu du)e^{\lambda dv} & xe^{\lambda dv} + \mu du \arrow[l, "d\tau"', color=red] & xe^{\lambda dv} \arrow[l, dashed]
\end{tikzcd}
$$

$$d\tau = (x + \mu du)e^{\lambda dv} - (xe^{\lambda dv} + \mu du)$$

$$
\begin{tikzcd}
x + \mu du \arrow[d] & & x \arrow[ll] \\
(x + \mu du)e^{\lambda dv} \arrow[r, "-d\tau", color=red] & xe^{\lambda dv} + \mu du \arrow[r] & xe^{\lambda dv} \arrow[u]
\end{tikzcd}
$$

$$0 = ((x + \mu du)e^{\lambda dv} - d\tau - \mu du)/e^{\lambda dv} - x$$

A picture: torsion is "tearing" from the flat



This "tearing" is a geometric manifestation of non-commutativity; while the flat is just the ACS plane.

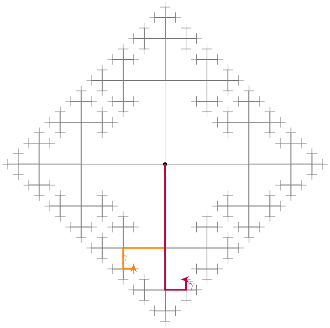


Figure: An abstract hyperbolic space

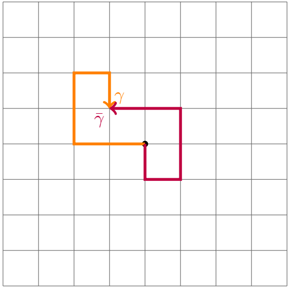


Figure: ACS space

In ACS, a path γ and its reversal $\bar{\gamma}$ always meet. ACS can be (u, v)-parameterized in which du and dv can be interpreted as addition and multiplication with weights e^v.

Consider $(u, v, a) \in \mathbb{R}^3$ with constants μ, λ.

$$\omega := \mu\, du + \lambda a\, dv, \qquad \alpha := da - \omega$$

Key properties:

- Contact:

$$d\omega = \lambda\, da \wedge dv, \quad d\alpha = -d\omega, \quad \alpha \wedge d\alpha = \mu\lambda\, du \wedge da \wedge dv \neq 0$$

  so α is a contact form when $\mu\lambda \neq 0$.
- Reeb field and distribution:

$$R = -(1/\mu)\, \partial_u, \qquad \mathcal{H} := \ker \alpha$$

- Horizontal lifts (basis of $\mathcal{H}$):

$$D_u := \partial_u + \mu\, \partial_a, \qquad D_v := \partial_v + \lambda a\, \partial_a, \qquad \alpha(D_u) = \alpha(D_v) = 0$$

- Natural units: with $\tilde{u} = \mu u$, $\tilde{v} = \lambda v$,

$$\alpha = da - d\tilde{u} - a\, d\tilde{v}$$

The expression differential δ is defined by

$$\delta a = \omega, \qquad \delta u = du, \qquad \delta v = dv,$$

and for any $F(u, v, a)$

$$\delta F = dF - (\partial_a F)\,\alpha = (D_u F)\,du + (D_v F)\,dv,$$

where

$$D_u F = F_u + \mu F_a, \qquad D_v F = F_v + \lambda a F_a, \qquad D_\theta = \cos\theta\, D_u + \sin\theta\, D_v.$$

Chain rules:

$$\delta \Phi(a) = \Phi'(a)\,\omega, \qquad \delta F(E_1, E_2) = \partial_1 F\,\delta E_1 + \partial_2 F\,\delta E_2.$$

Curvature and non-commutativity:

$$[D_u, D_v] = \mu\lambda\,\partial_a, \qquad \delta^2 F = \mu\lambda(\partial_a F)\,du\wedge dv, \qquad \delta^2 a = \mu\lambda\,du\wedge dv.$$

Compatibility and circulation:

$$(d\omega)^*\big|_{\alpha=0} = \mu\lambda\,du\wedge dv, \qquad \oint_{\partial\Sigma}\omega = \iint_{\Sigma} d\omega = \mu\lambda\iint_{\Sigma} du\wedge dv.$$

Quick rules:

$$\delta(a^n) = na^{n-1}\omega, \quad \delta(\ln a) = \tfrac{\omega}{a}, \quad \delta(e^a) = e^a\omega, \quad \delta(\sin a) = \cos a\,\omega, \quad \delta(\cos a) = -\sin a\,\omega.$$

Rectification and flow:

$$y = \arcsin\left(\frac{\lambda a}{\mu}\right), \quad \|\nabla y\| = \lambda, \quad \frac{da}{ds} = D_\theta a = \mu\cos\theta + \lambda a\sin\theta.$$

Natural units: $\tilde{u} = \mu u$, $\tilde{v} = \lambda v$. Define scaled powers

$$B_n(a, v) := e^{-n\tilde{v}} a^n \quad (n \in \mathbb{N}).$$

Let

$$\mathcal{B} := \Big\{ \sum_{n=0}^{N} P_n(u, v, e^{\tilde{v}})\, B_n(a, v) \text{ (finite)} \Big\}.$$

Closure Theorem. $\mathcal{B}$ is closed under the mixed calculus generated by

$$D_u = \partial_u + \mu\, \partial_a, \qquad D_v = \partial_v + \lambda a\, \partial_a.$$

Rules:

$$D_u(PB_n) = (\partial_u P)B_n + \mu n\, e^{-\tilde{v}} P\, B_{n-1}, \quad D_v(PB_n) = (\partial_v P)B_n.$$

If P is independent of u,

$$D_u^{-1}\big(P(v, e^{\tilde{v}})\, B_n\big) = \frac{e^{\tilde{v}} P(v, e^{\tilde{v}})}{\mu(n+1)}\, B_{n+1}.$$

In general, define $Q_n^{(0)} = \dfrac{e^{\tilde{v}}}{\mu(n+1)} P_n$ and set $G^{(0)} = \sum_n Q_n^{(0)} B_{n+1}$. Then $F - D_u G^{(0)} = -\sum_n (\partial_u Q_n^{(0)}) B_{n+1}$. Repeat for finitely many steps.

Example. For $F = a^3 e^{\lambda v} = e^{4\tilde{v}} B_3$,

$$D_u^{-1} F = \frac{e^{\tilde{v}}}{4\mu} B_4 = \frac{e^{\lambda v}}{4\mu} a^4, \quad D_u\left(\frac{e^{\lambda v}}{4\mu} a^4\right) = a^3 e^{\lambda v}.$$

Conclusion: a new calculus for mixed operations

We have shown that:

- Mixed additive/multiplicative change admits a contact geometry on (u, v, a) with $\alpha = da - \omega$, $\omega = \mu\, du + \lambda a\, dv$; $\mathcal{H} = \ker \alpha$ has basis $\{D_u, D_v\}$ and α is contact for $\mu\lambda \neq 0$.
- The expression differential δ projects d to $\ker \alpha$: $\delta F = dF - (\partial_a F)\alpha$, with chain rules and directional synthesis via D_θ.
- Non-commutativity/curvature are encoded by $[D_u, D_v] = \mu\lambda\, \partial_a$ and $\delta^2 F = \mu\lambda(\partial_a F)\, du \wedge dv$; circulation-area and de Rham compatibility hold.
- Legendrian flow obeys $\frac{da}{ds} = \mu \cos\theta + \lambda a \sin\theta$; rectification $y = \arcsin(\lambda a/\mu)$ yields $\|\nabla y\| = \lambda$ and stabilizes numerics; natural units $(\tilde{u}, \tilde{v})$ simplify formulas.
- An affine–Appell basis $B_n(a, v) = e^{-n\tilde{v}} a^n$ is closed under D_u, D_v, enabling explicit differentiation/integration and constructive antiderivatives for mixed expressions.

A New Atlas for Complex Analysis?

Is complex analysis a special case of a more general theory?

This stems from the fact that multiplication by -1 is involutive, i.e., $(-1)^2 = 1$. We therefore need an infinitesimal generator that connects -1 and 1 continuously.

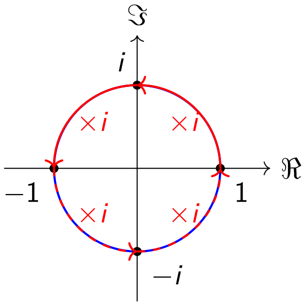


Figure: i as a generator: multiplication by i rotates by 90°, so $1 \xrightarrow{\times i} i \xrightarrow{\times i} -1$.

Setup (over $\mathbb{C}$): contact structure and affine–Appell chart

Complexification rule. Coordinates $(u, v) \in \mathbb{R}^2$; values and parameters $a, \mu, \lambda \in \mathbb{C}$. All operators act $\mathbb{C}$-linearly.

Contact data and horizontal fields

$$\omega := \mu\, du + \lambda a\, dv, \qquad \alpha := da - \omega,$$

$$D_u := \partial_u + \mu\, \partial_a, \qquad D_v := \partial_v + \lambda a\, \partial_a, \qquad \delta F = (D_u F)\, du + (D_v F)\, dv.$$

Natural units and affine–Appell coordinate

$$\tilde{u} := \mu u, \quad \tilde{v} := \lambda v, \qquad s := a\, e^{-\tilde{v}} \;\Rightarrow\; D_v s = 0,\; D_u s = \mu\, e^{-\tilde{v}}.$$

We write $B_n := s^n$ $(n \in \mathbb{N})$ and use an EL-closed coefficient ring $\mathcal{R}_{\mathrm{EL}}$ generated by $u, v, e^{\pm\tilde{v}}$ and closed under $+, \times, \partial_u, \partial_v, \exp, \log$ (where defined).

Arithmetic-holomorphic (over $\mathbb{C}$). For $h = f + ig$ with $f, g : (u, v, a) \mapsto \mathbb{R}$, define

$$\bar{D} := \tfrac{1}{2}(D_u + iD_v), \qquad \text{AEG-holomorphic} \iff \bar{D}h = 0.$$

Equivalently, the **AEG–CR equations**

$$D_u f = D_v g, \qquad D_v f = -D_u g. \tag{16}$$

Geometric consequences.

- *Conformality and equal norms*: $(D_u f, D_u g) \perp (D_v f, D_v g)$ and $(D_u f)^2 + (D_u g)^2 = (D_v f)^2 + (D_v g)^2$.
- *Classical limit*: if f, g are independent of a, then $D_u = \partial_u$, $D_v = \partial_v$, yielding the usual CR equations on (u, v).
- *Rigidity*: if $f = f(a)$ and $g = g(a)$, the only solutions are constants.

Theorem (local expansion and uniqueness). Let $h = f + ig$ be C^1 on a neighborhood and satisfy $\bar{D}h = 0$. Then there exist unique coefficients $A_n(u, v) \in \mathcal{R}_{\mathrm{EL}}$ (real-analytic in (u, v)) such that

$$h(u, v, a) = \sum_{n \geq 0} A_n(u, v)\, s^n, \qquad s = a\, e^{-\tilde{v}}.$$

The coefficients obey the triangular recursion

$$\frac{\mu}{2}\, e^{-\tilde{v}}\, (m + 1)\, A_{m+1}(u, v) + \partial_{\bar{\zeta}} A_m(u, v) = 0, \quad \partial_{\bar{\zeta}} := \tfrac{1}{2}(\partial_u + i\partial_v).$$

Conversely, any convergent series with this recursion solves $\bar{D}h = 0$.
Meromorphic extension. Allowing a finite principal part $h = \sum_{n \geq N} A_n s^n$ $(N \in \mathbb{Z})$ yields AEG–meromorphic functions (poles at $s = 0$).

For $F = \sum_{n \geq 0} A_n s^n$:

$$D_u F = \sum (\partial_u A_n)\, s^n + \mu e^{-\tilde{v}} \sum_{n \geq 1} n A_n\, s^{n-1}, \qquad D_v F = \sum (\partial_v A_n)\, s^n.$$

Indefinite integration along u. Find $G = \sum B_n s^n$ with $D_u G = F$:

$$B_0 \text{ free}, \qquad B_{k+1} = \frac{A_k - \partial_u B_k}{\mu e^{-\tilde{v}}(k+1)} \quad (k \geq 0).$$

Algebraic closure.

- Closed under addition and the "AEG complex product" $(f, g) \odot (\tilde{f}, \tilde{g}) = (f\tilde{f} - g\tilde{g},\ f\tilde{g} + \tilde{f}g)$.
- Composition with a classical holomorphic function of $\zeta = u + iv$ preserves AEG-holomorphicity.
- Laurent series in s implement the AEG–meromorphic class (residues at $s = 0$).

Write $F = (f, g)$ and $|F'|^2 := (D_u f)^2 + (D_u g)^2 = (D_v f)^2 + (D_v g)^2$. Then

$$d\omega = \frac{\mu\lambda}{|F'|^2}\, df \wedge dg. \tag{17}$$

The area density rescales by the Jacobian $|F'|^{-2}$, while the total integral $\int d\omega$ is invariant (expression-conformal coordinates).

Fiberwise complex analysis in s. Fix (u, v); then $s \in \mathbb{C}$ and $h(\cdot) = \sum A_n s^n$ is holomorphic in s. Hence the classical Cauchy machinery applies fiberwise:

$$h(s_0) = \frac{1}{2\pi i} \oint_\gamma \frac{h(s)}{s - s_0}\, ds, \qquad \oint_\gamma h(s)\, ds = 2\pi i \cdot A_{-1}(u, v),$$

for any loop γ encircling s_0 (or 0 for the residue statement).

Constructor (seed $\Rightarrow$ AEG-holomorphic). Pick any real-analytic seed $A_0(u, v)$; define recursively

$$A_{m+1}(u, v) = -\frac{2\, e^{\tilde{v}}}{\mu\,(m+1)}\, \partial_{\bar{\zeta}} A_m(u, v), \qquad m \geq 0,$$

and set $h = \sum_{n \geq 0} A_n s^n$ with $s = a\, e^{-\tilde{v}}$. Then $\bar{D}h = 0$. If A_0 is classical holomorphic in ζ, all higher $A_{n \geq 1}$ vanish (classical limit).

Explicit nontrivial example. Take $A_0(\zeta, \bar{\zeta}) = \bar{\zeta}$; then

$$A_1 = -\tfrac{2}{\mu} e^{\tilde{v}}, \quad A_2 = \tfrac{i\lambda}{\mu^2} e^{2\tilde{v}}, \quad A_3 = \tfrac{2\lambda^2}{3\mu^3} e^{3\tilde{v}}, \ \ldots$$

so

$$h(u, v, a) = \bar{\zeta} - \frac{2}{\mu}\, a + \frac{i\lambda}{\mu^2}\, a^2 + \frac{2\lambda^2}{3\mu^3}\, a^3 + \cdots$$

is AEG-holomorphic (convergent in a neighborhood controlled by $|a|$ and the majorant of A_0).

Conclusion: A new context for holomorphic and analytic functions

We have shown that:

- AEG–Cauchy–Riemann: $\bar{D} = \frac{1}{2}(D_u + iD_v)$, with $\bar{D}h = 0$ defining arithmetic holomorphicity; conformality and the classical limit are recovered, while rigidity appears for functions depending only on a.
- AEG–Weierstrass expansion: any local AEG–holomorphic h admits $h = \sum_{n\geq 0} A_n(u, v)\, s^n$ with triangular recursion; Laurent series in s yield meromorphic extensions.
- Computation rules: explicit formulas for D_u, D_v on series and a constructive scheme for indefinite integration along u; closure under the AEG complex product and composition with classical holomorphic maps in $\zeta = u + iv$.
- Geometry and analysis align: $d\omega = \frac{\mu\lambda}{|F'|^2} df \wedge dg$ encodes conformal area density; fiberwise in s, classical Cauchy integral and residue calculus apply.

A Key to the Nonlinear Maze?

Can AEG help us navigate the complexities of nonlinear systems?

Why is this possible?

- The multiplicative dimension of AEG inherently encodes nonlinearity.
- The non-commutativity of operations is native to the AEG framework.
- A spatiotemporal view of complexity clarifies slow and fast variables in a synergetics setting.

In what follows, we show that AEG's Accumulative Commutative Space (ACS) functions as a minimalist RG flow.

ACS and RG: commutative coordinates and scale

- Accumulative Commutative Space (ACS) assigns to any arithmetic path γ the coordinates

$$A_\gamma = \sum_k \mu_k, \qquad M_\gamma = \sum_k \lambda_k,$$

where μ_k (additive) and λ_k (log-multiplicative) are the operation parameters along γ.

- Interpretation:
    - A captures the *net additive load* accumulated by the computation.
    - M encodes the *evolutionary scale* (log-scale) accumulated by multiplicative steps.
- Minimal RG perspective: varying M probes effective dynamics (driven by addition and local flows) at different scales. ACS provides a commutative plane to compare paths across scales.

Example I: Percolation RG ($R(p) = 3p^2 - 2p^3$)

- Polynomial form $R(p) = Yp^N + Xp^{N-1}$ with $(X, Y, N) = (3, -2, 3)$.
- ACS coordinates (canonical AEG path):

$$A_R = X + Y = 1, \qquad M_R(p) = (N-1)\ln p = 2\ln p.$$

- Fixed points in ACS:

$$p^* = 0: \ (A = 1, \ M \to -\infty), \quad p^* = \tfrac{1}{2}: \ (A = 1, \ M = -2\ln 2),$$

$$p^* = 1: \ (A = 1, \ M = 0).$$

- A torsion instance (path-dependent) $\mathcal{T}_R(p) = -\ln p\,(8p^{-1} + 3)$ illustrates finite, nonzero torsion at the critical point and vanishing torsion at $p = 1$.

Example II: Logistic map ($R(x) = rx(1-x)$, $r = 3.2$)

- $R(x) = (-r)x^2 + (r)x$ with $N = 2$, coefficients $(c_2, c_1, c_0) = (-r, r, 0)$.
- ACS coordinates (Horner path):

$$A_R(r) = r, \qquad M_R(x) = 2\ln x.$$

- Stable 2-cycle at $r = 3.2$: $x_a \approx 0.513045$, $x_b \approx 0.799455$.

$$M_R(x_a) \approx -1.3348, \quad M_R(x_b) \approx -0.4476.$$

- Second iterate $R^{(2)}$: $A_{R^{(2)}} = r^3 = 32.768$, $M_{R^{(2)}}(x) = 4\ln x$. The points x_a, x_b are fixed points of $R^{(2)}$, reflecting a deeper scale M.

Takeaways: ACS as a minimalist RG flow

- ACS provides (A, M) coordinates that separate dynamical content (addition) from evolutionary scale (multiplication).
- The AEG flow encodes scale dependence directly via λa, with M acting as evolutionary time; varying M reveals effective behavior at different scales.
- Arithmetic torsion integrates scale (e^M) into a geometric invariant, yielding phase-sensitive signatures and principled path comparisons.
- Across examples (percolation, logistic), fixed points and cycles appear as specific loci in ACS; M quantifies scale while A captures net additive load.

We've journeyed through five possibilities, each a "small, colorful stone" representing a different facet of Arithmetic Expression Geometry.

- **Ochre:** Non-commutativity implies exponential complexity, geometrically realized by the volume of hyperbolic space.
- **Lapis:** A unified framework where computation, geometry, and analysis converge, linking expressions to geometric flows and topological invariants.
- **Malachite:** A neo-calculus on a contact structure ($\alpha = da - \omega$) for mixed additive and multiplicative dynamics.
- **Quartz:** A generalization of complex analysis to an "arithmetic-holomorphic" framework where $\bar{D}h = 0$.
- **Obsidian:** A minimalist, RG-like flow via Accumulative Commutative Space (ACS) to analyze scale in non-linear systems.

These five stones suggest that the geometry of arithmetic offers a new language, unifying our understanding of complexity, analysis, and dynamics.

Acknowledgments

We thank

- discussions with Yue Chen, Yangzhou Liu and Bing Yuan;
- vast contributions by ChatGPT and Gemini AI;
- support from my family and friends over the years.

Thanks