Arithmetic Expression Geometry

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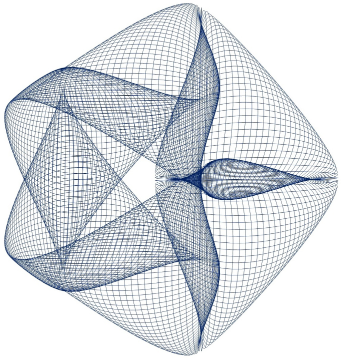
1. Background
2. Our idea and exploration

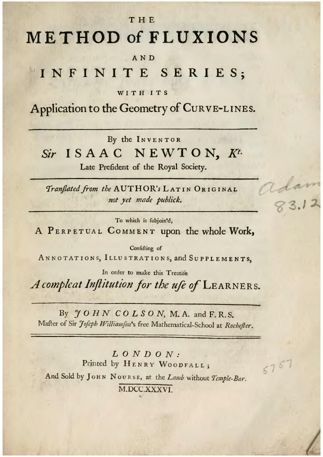
Background



Two methods to describe a small change over time:

- by quantity: adding a small near-zero amount of quantity
- by ratio: multiplying a near-unit ratio

Traditional calculus is based on the first method, Riemann integral is additive. We can use functions exp and log to convert between the two methods.



THE
METHOD of FLUXIONS
AND
INFINITE SERIES;
WITH ITS
Application to the Geometry of CURVE-LINES.

By the INVENTOR
Sir ISAAC NEWTON, Kt.
Late Preſident of the Royal Society.

Tranſlated from the AUTHOR's LATIN ORIGINAL
not yet made publick.

To which is ſubjoin'd,
A PERPETUAL COMMENT upon the whole Work,
Conſiſting of
ANNOTATIONS, ILLUSTRATIONS, and SUPPLEMENTS,
In order to make this Treatiſe
A compleat Inſtitution for the uſe of LEARNERS.

By *JOHN COLSON*, M.A. and F.R.S.
Maſter of Sir *Joſeph Williamſon*'s free Mathematical-School at *Rocheſter*.

LONDON:
Printed by HENRY WOODFALL;
And Sold by JOHN NOURSE, at the *Lamb* without *Temple-Bar*.
M.DCC.XXXVI.

Figure: Method of Fluxions

Figure: Vito Volterra

Matrix-valued non-commutative derivative and integration, left and right

- $\frac{d}{dx}A(x) = \lim_{\Delta x \to 0} \frac{A(x+\Delta x)A^{-1}(x)-I}{\Delta x}$
- $A(x)\frac{d}{dx} = \lim_{\Delta x \to 0} \frac{A^{-1}(x)A(x+\Delta x)-I}{\Delta x}$
- $\prod_a^b (I + A(x)dx) = \lim_{\nu(P)\to 0} \prod_{i=m}^{1} (I + A(\xi_i))$
- $(I + A(x)dx) \prod_a^b = \lim_{\nu(P)\to 0} \prod_{i=1}^{m} (I + A(\xi_i))$

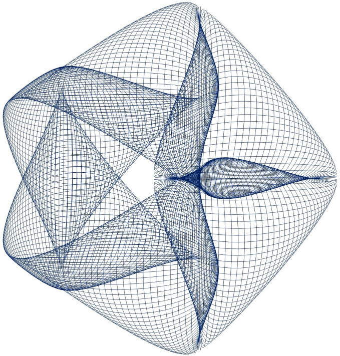
An interesting formula connect product integration and normal additive integration

$\prod_a^b (I + A(x)dx) = I + \int_a^b A(x)dx + \int_a^b \int_a^x A(x)A(y)dydx + \cdots$

How about mixing up additive and multiplicative steps?

Question: How about mixing additive and multiplicative steps up in one process?
We can get a first order non-homogeneous differential equation

$$\frac{dx}{dt} = (1 + f(t))x + g(t)$$



The famous example of word2vec

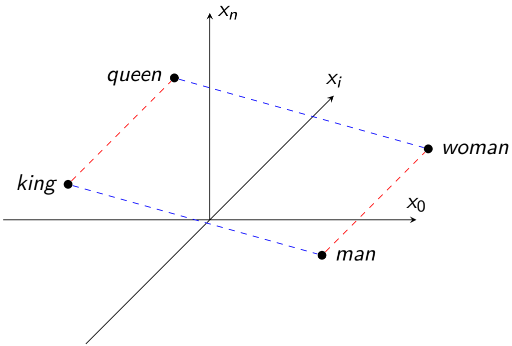


Figure: regulairty of word2vec

$$(\alpha + 1) \times 2 \neq \alpha \times 2 + 1$$

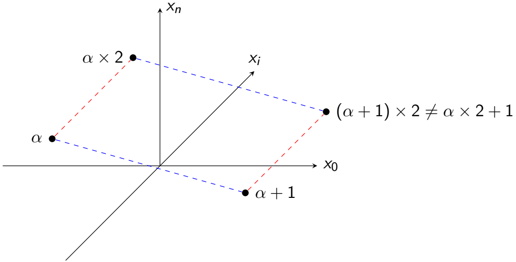
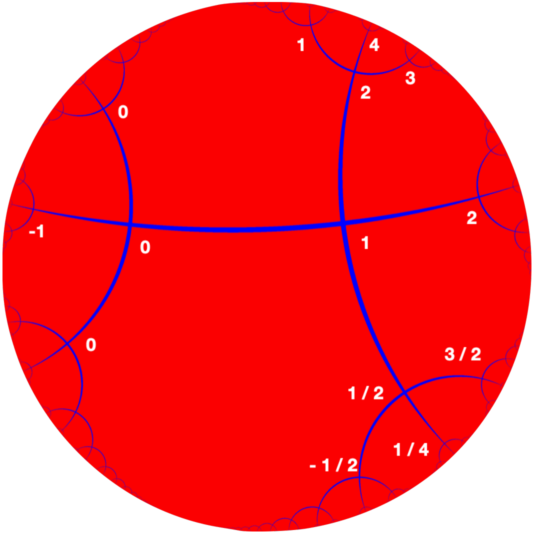


Figure: contradiction of numbers in Euclidean space

One arrangement in hyperbolic space



Suppose we have a base point a_0, and we step a small distance away from a_0.
Addition first

$$a_\delta = (a_0 + \mu\epsilon \cos\theta) e^{\lambda\epsilon \sin\theta}$$

Multiplication first

$$a_\delta = a_0 e^{\lambda\epsilon \sin\theta} + \mu\epsilon \cos\theta$$

Both formula can be simplified to the same result:

$$a_\delta = a_0 + \epsilon(a_0\lambda \sin\theta + \mu\cos\theta)$$

Then, we have the following equation:

$$\frac{1}{\delta}(a_\delta - a_0) = \frac{\epsilon}{\delta}(\mu\cos\theta + a_0\lambda\sin\theta)$$

When both δ and ϵ are towards zero, we get da/dt, and hence

$$\frac{da}{dt} = u(\mu\cos\theta + a\lambda\sin\theta)$$

Or, we can change it to another form

$$\frac{da}{ds} = \mu\cos\theta + a\lambda\sin\theta \tag{1}$$

The flow equation can be solved formally

$$\frac{da}{ds} = \mu \cos\theta + a\lambda \sin\theta$$

and the formal solution is

$$a = a_0 e^{\lambda s \sin\theta} + \frac{\mu}{\lambda}(e^{\lambda s \sin\theta} - 1)\cot\theta \tag{2}$$

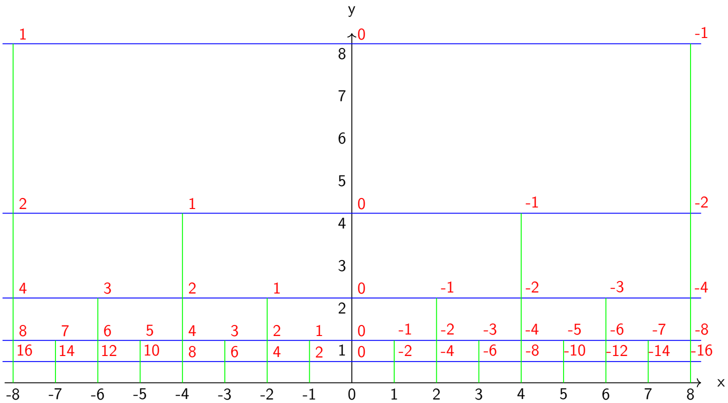
$$a = a_0 e^{\lambda s \sin\theta} + \mu s \cos\theta + \frac{\mu}{2\lambda}\sin 2\theta\left(\frac{\lambda^2 s^2}{2!} + \frac{\lambda^3 s^3}{3!}\sin\theta + \frac{\lambda^4 s^4}{4!}\sin^2\theta + \cdots\right)$$

$$a = a_0 e^{\lambda s \sin\theta} + \mu s \cos\theta + \frac{\mu}{2\lambda}\Psi(s)\sin 2\theta$$

So $\theta = 2k\pi$ encode addition, $\theta = 2k\pi + \frac{\pi}{2}$ encode multiplication, and $\theta = 2k\pi + \pi$ encode subtraction, and $\theta = 2k\pi + \frac{3\pi}{2}$ encode division.

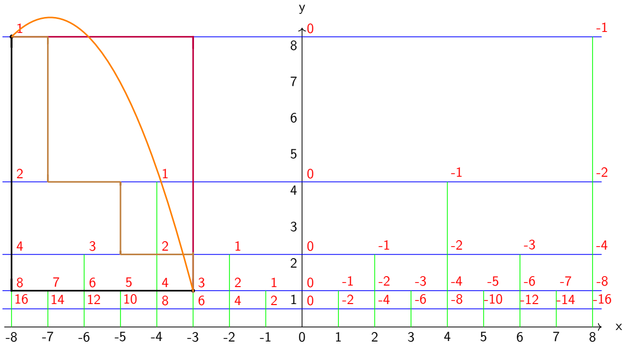
Another arrangement in hyperbolic space

$$a = -\frac{x}{y}$$

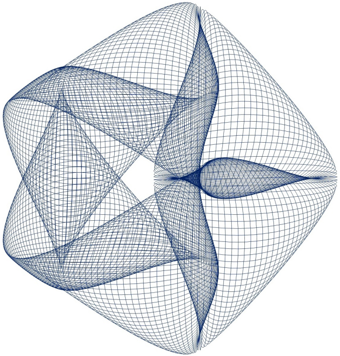


Encoding threadlike expressions as paths

- black line $1 \times 8 - 5 = 3$
- purple line $(1 - \frac{5}{8}) \times 8 = 3$
- orange line: a speical integration



1. Basic concepts
2. Case study: $\mathfrak{E}_1 space$
3. Unsolved problems



Giving an arithmetic expression, we can parse it into a syntax tree. For example, the expression

$$(((((1 \times 2) \times 2) - 1) \times (2 + 1)) - 6) \tag{3}$$

and the parsed syntax tree

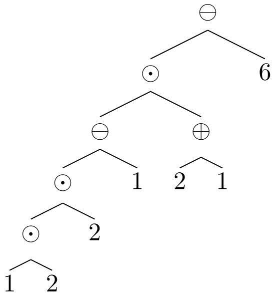


Figure: a tree representation of an arithmetic expression

Definition

An arithmetic expression a over $\mathbb{Q}$ is a structure given by the following production rules:

$$\begin{aligned} a &\longleftarrow x \\ a &\longleftarrow (a + a) \\ a &\longleftarrow (a - a) \\ a &\longleftarrow (a \times a) \\ a &\longleftarrow (a \div a) \end{aligned} \tag{4}$$

where $x \in \mathbb{Q}$, and we denote this as $a \in \mathbb{E}[\mathbb{Q}]$.

We can define evaluation $\nu(a)$ of a recursively as follows:

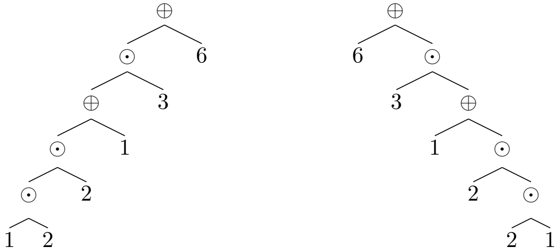
- Constant leaf: for any $x \in \mathbb{Q}$, $\nu(x) = x$.
- Compositional node by $+$: For any $(a + b)$, $\nu((a + b)) = \nu(a) + \nu(b)$.
- Compositional node by $-$: For any $(a - b)$, $\nu((a - b)) = \nu(a) - \nu(b)$.
- Compositional node by $\times$: For any $(a \times b)$, $\nu((a \times b)) = \nu(a)\nu(b)$.
- Compositional node by $\div$: For any $(a \div b)$, if $\nu(b) \neq 0$, then $\nu((a \div b)) = \nu(a)/\nu(b)$.

Generally, the evaluation order of the arithmetic expression is not unique though the result is decided.

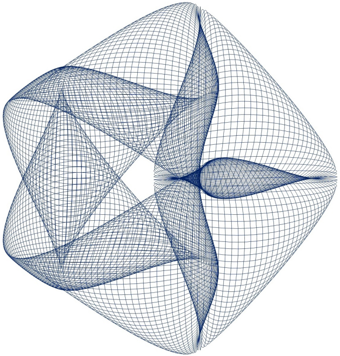
Threadlike expressions

Right-expanded and left-expanded threadlike expressions

$$(((((1 \times 2) \times 2) + 1) \times 3) + 6) \qquad (6 + (3 \times (1 + (2 \times (2 \times 1)))))$$



The evaluation order of threadlike expressions is unique. We take left-expanded threadlike expressions as the standard form.



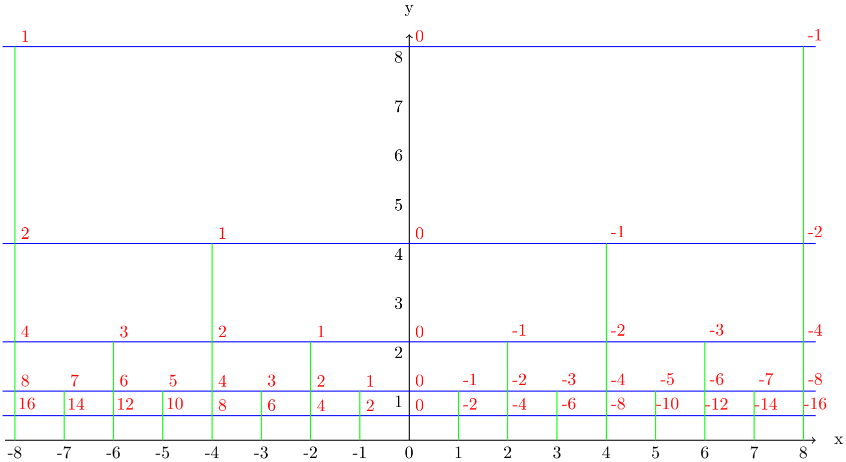
The hyperbolic space equipped with the metric

$$ds^2 = \frac{1}{y^2}(\frac{dx^2}{\mu^2} + \frac{dy^2}{\lambda^2})$$

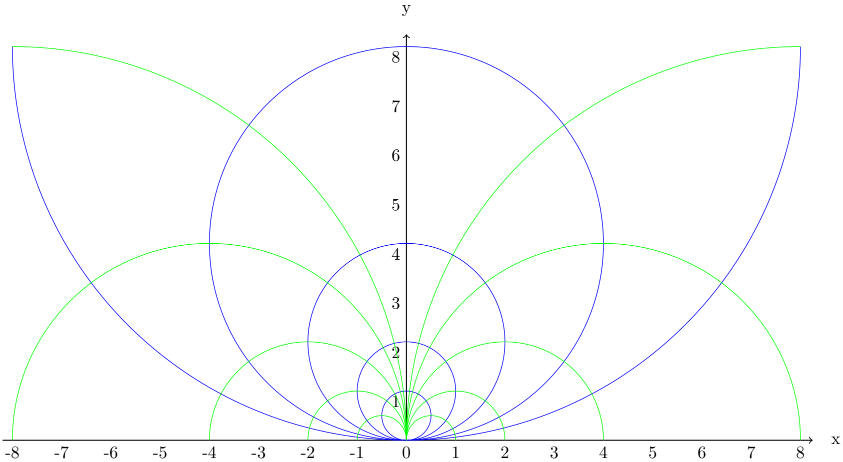
and a scalar field

$$a = -\frac{x}{y}$$

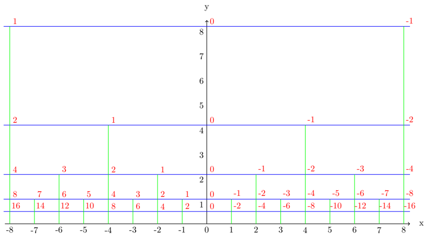
The flow equation is satisfied.

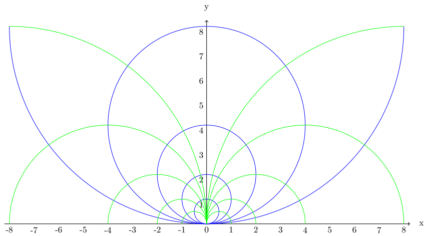

y
1 0 -1
8
7
6
5
2 1 0 -1 -2
4
3
4 3 2 1 0 -1 -2 -3 -4
2
8 7 6 5 4 3 2 1 0 -1 -2 -3 -4 -5 -6 -7 -8
16 14 12 10 8 6 4 2 1 0 -2 -4 -6 -8 -10 -12 -14 -16
-8 -7 -6 -5 -4 -3 -2 -1 0 1 2 3 4 5 6 7 8
x

Another grid on $\mathfrak{E}_1$ space



Möbius transformation





$$z \mapsto -\frac{1}{z}$$

In our setting, $A = \frac{1}{\mu y}$ and $B = \frac{1}{\lambda y}$:

$$\Delta f = y^2 \left(\mu^2 \frac{\partial^2 f}{\partial x^2} + \lambda^2 \frac{\partial^2 f}{\partial y^2} \right)$$

And for the function $f = -\frac{x}{y}$, we have

$$\Delta f = -\frac{2\lambda^2 x}{y} = 2\lambda^2 f$$

So, we reach the conclusion that the function $f = -\frac{x}{y}$ is a eigenfunction of the Laplacian with eigenvalue $2\lambda^2$.

Two approaches towards arithmetic expression space?

We have several relevant structures:

- $E(F), (H, a), (Path, Integ)$

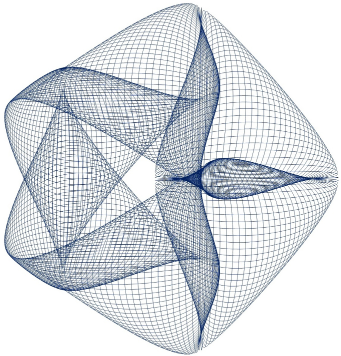
A constructive approach: from well-defined $E(Q)$, we introduce a proper topology and metric to form a space with a compatible condition

1. Convergence geometrically can lead to convergence of arithmetic evaluation
2. The arithmetic evaluation can be extended to the whole topological space continuously

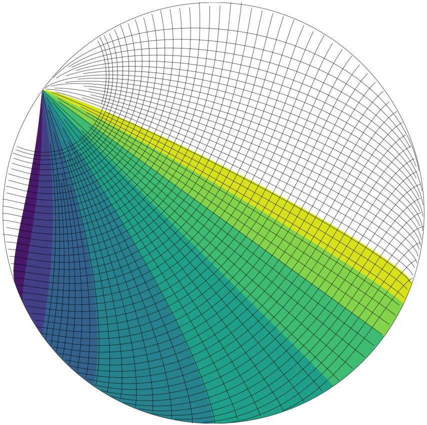
An interpretative approach: we can interpret a special zigzag path of a well-defined space as an arithmetic expression.

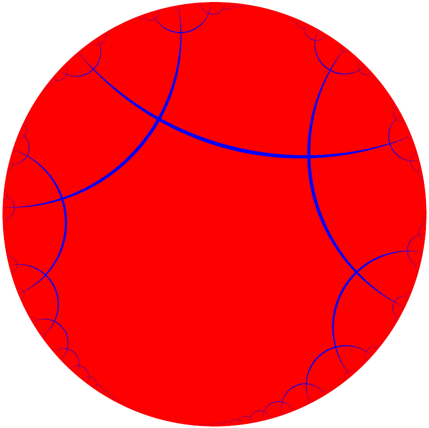
1. Bijective: every threadlike arithmetic expression can be interpreted as a zigzag path, and versa vice

Unsolved problems



$\mathfrak{E}_1$ is too rigid, can we find a space that is more flexible?





Existence theorems

- Local metric existence theorem
- Proper assignment existence?

Local structure: totally decided by the flow equation.
Classification of the global structure

Eigenfunction of Laplacian in $\mathfrak{E}_1$ space might not be a special case.

We cannot make $\times -1$ compatible with the flow equation in the real number case, but we can do it in the complex number case easily with the imaginary unit i. complex arithmetic expression space is mandatory in the theory.

Repeating one operation to generate operation in next order

Order	Operation	Notion
0	Succession	$a + 1$
1	Addition	$a + b$
2	Multiplication	$a \times b$
3	Exponentiation	a^b
4	Tetration	$a[4]b$
5	Pentation	$a[5]b$
6	Hexation	$a[6]b$

Can we compose high dimensional AEG space by hyper-operations?

1. Complex differentiable
2. Analytic
3. Holomorphic
4. Conformal
5. Maximum modulus principle

The Cauchy-Riemann condition

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

setup the compatibility between the complex structure and the differential structure, which means any path approaching a point should lead to the same derivative.

$$\frac{\delta f}{\delta z} = \frac{\delta u + i\delta v}{\delta x + i\delta y}$$

$$\left.\frac{\delta f}{\delta z}\right|_{\delta y=0} = \frac{\delta u}{\delta x} + i\frac{\delta v}{\delta x}$$

$$\left.\frac{\delta f}{\delta z}\right|_{\delta x=0} = -i\frac{\delta u}{\delta y} + \frac{\delta v}{\delta y}$$

A function is holomorphic if it is complex differentiable at every point in its domain.

$$f(z) = u(x, y) + iv(x, y)$$

u and v are a pair of harmonic functions, which satisfy the Laplace equation, and are conjugate to each other.

A function is analytic if and only if its Taylor series at point z_0 converges to the function in some neighborhood of z_0 for every z_0 in its domain.

$$f(z) = \sum_{n=0}^{\infty} a_n \, (z - z_0)^n = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots$$

A function f is holomorphic in a domain and continuous on the boundary of the domain. For any point z in the domain, C is a closed rectifiable curve which has winding number one around z, and C is contained in the domain, then we have

$$f(z) = \frac{1}{2\pi i} \oint_C \frac{f(\zeta)}{\zeta - z} d\zeta$$

We can use boundary value to determine the value of the function in the domain.

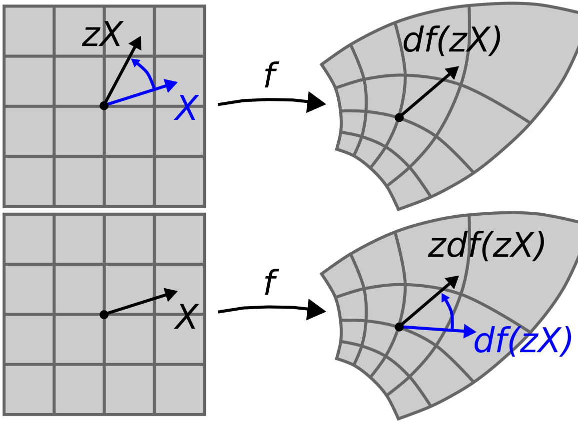


image credit: https://en.wikipedia.org/wiki/Cauchy-Riemann_equations

The modulus of a holomorphic function cannot exhibit a strict maximum that is strictly within its domain.

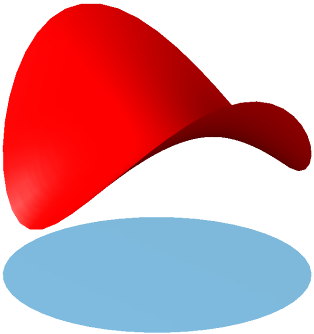
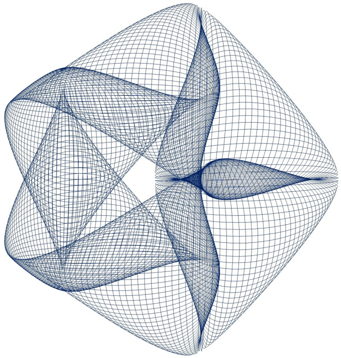


image credit: https://en.wikipedia.org/wiki/Maximum_modulus_principle

Future directions: geometry, analysis and computation

1. Geometry
2. Analysis
3. Computation



We scale up the step size
For one step, we have

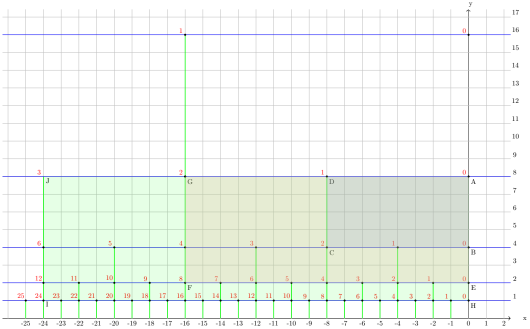
$$(x+1)\times 2-(x\times 2+1)=1 \qquad (5)$$

Extending this to two steps, we encounter a different situation:

$$(x+2)\times 4-(x\times 4+2)=6 \qquad (6)$$

And for three steps, the pattern continues:

$$(x+3)\times 8-(x\times 8+3)=21 \qquad (7)$$



$$d\tau = (a_0 + \mu du)e^{\lambda dv} - (a_0 e^{\lambda dv} + \mu du)$$

$$d\tau = \mu\lambda dudv$$

and because

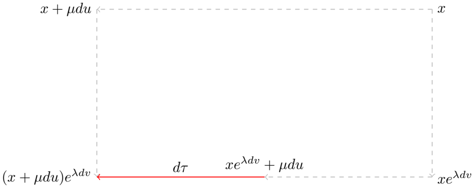
$$dS = \sqrt{A^2 B^2 - F^2} dudv$$

$$\frac{d\tau}{\mu\lambda} = \frac{dS}{\sqrt{A^2 B^2 - F^2}} \tag{8}$$

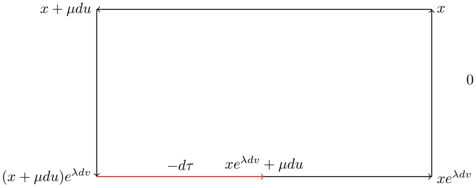
Curvature?

Arithmetic torsion is a quantity defined as a measure to reflect how the generators are non-commutative. It is defined on path, but when we scale up the step size, we find that torsion is related to the size of an area.
Similar case can be found in the curvature of a surface, stated by the Gauss-Bonnet theorem.
Can we find a connection between arithmetic torsion and curvature?

A broken clue?

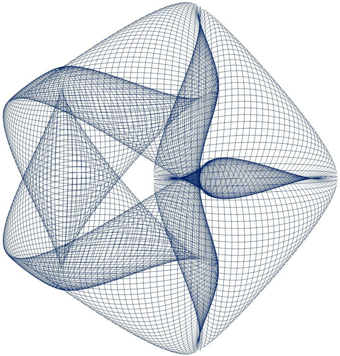


$$d\tau = (x + \mu du)e^{\lambda dv} - (xe^{\lambda dv} + \mu du)$$



$$0 = ((x + \mu du)e^{\lambda dv} - d\tau - \mu du)/e^{\lambda dv} - x$$

Analysis



Path integration?

We may extends Riemann integral to a path integral. What properties should the path integral have? Is the eigenfunction of Laplacian in $\mathfrak{E}_1$ a special case?

The picture of flow equation

$$\frac{da}{ds} = \mu \cos\theta + a\lambda \sin\theta$$

is that any path escape from a point leads to a different value. If we reverse the direction of the path, we have another interpretation of the flow equation. It can be also interpreted as a differentiability condition, any path approaching a point should lead to the same derivative, which is similar to the Cauchy–Riemann condition in complex analysis.

For any a, we have a θ make $\frac{da}{ds}$ positive

$$\frac{da}{ds} = \mu \cos\theta + a\lambda \sin\theta$$

This means a maximum value principle is satisfied, which is similar to the maximum modulus principle in complex analysis.

In $\mathfrak{E}_1$ space, assignment a is eigenfunction of Laplacian with eigenvalue $2\lambda^2$. When $\lambda = 0$, it is pure additive case, and the curvature is zero also, and a is harmonic.

Formulate our systems as a tuple of

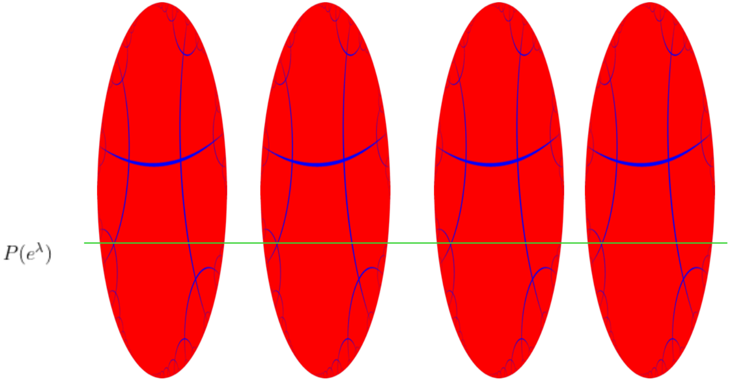
- $E(F), (H, a), (Path, Integ)$

Here we have

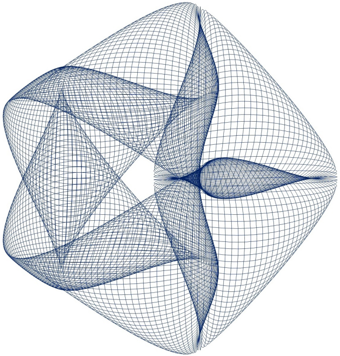
- $E(F)$: Expressions over a field F
- (H, a): A scalar field "assignment" a on a space H
- $(Path, Integ)$: all paths can be interpreted as an integral

Can we find more examples? Does complex analysis belong to this structure? We conjecture that complex analysis is a 1-dimensional additive AEG theory.

Tube structure?



Real version and complex version. A polynomial as a fiber across the AEG spaces with different generators.

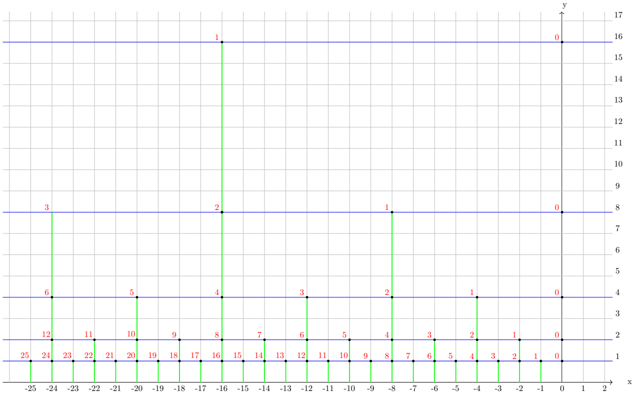


Flow perspectives

- Function as flow
- Computation as flow, Effective flow

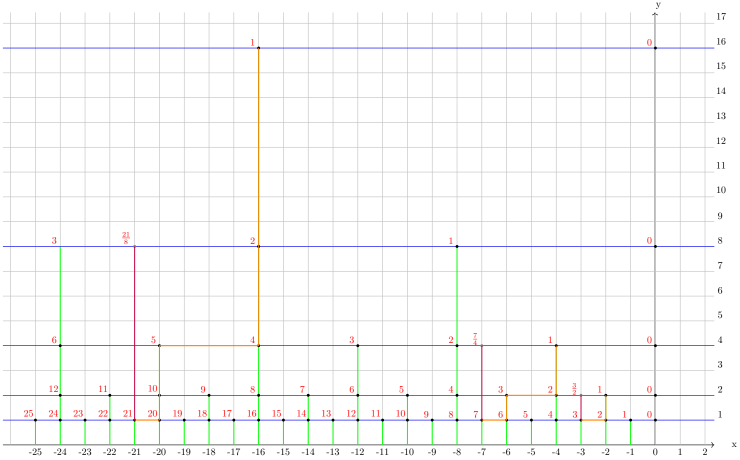
$$\begin{array}{ccc} H & \xrightarrow{\;I\;} & H \\ \nu\big\downarrow & & \big\downarrow\nu \\ R & \xrightarrow[\;k\;]{} & R \end{array}$$

Binary numeral as effective flow



We have

$$\frac{3}{2}_{10} = 1.1_2,\ \frac{7}{4}_{10} = 1.11_2 \text{ and } \frac{21}{8}_{10} = 10.101_2.$$



Using multiplication as a case study, we should focus on the following questions:

- How to define the flow of multiplication?
- How to measure the time complexity of multiplication flow?
- How to measure the space complexity of multiplication flow?

Space-time complexity?

The trade-off between space and time complexity is common in algorithm design, but the conversion between space and time complexity means they are the same thing from a conceptual perspective. We need study the trace of a computation process. AEG provides a testbed for this study.

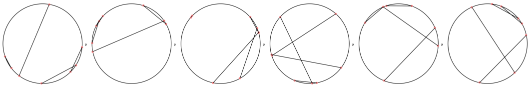
1. Grothendieck on triviality
2. A kid's problem

Alexander Grothendieck once emphasized the triviality in mathematics in his letter:

The difficulty of bringing new concepts out of the dark ...

A kid's problem

Cake-cutting is a classic problem and studied in mathematics, computer science, economics and political science, and it is taught in primary school.
One day, my kid asked me a question: "Why the more pieces I cut the more unfairness I get, if we use as less cut as possible?"



We may introduce "area entropy" and "edge entropy" to measure the fairness of a cake-cutting, but what is the connection between the entropy and the number of cuts? What is the relationship between the "area entropy" and "edge entropy"?

Open your eyes as a kid, keep curiosity alive and mind open, and you will find the beauty of mathematics everywhere.
Thank you!